THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 11 Problems
15th April 2024

- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. For each of the following polynomial over the stated rings, decide whether it is irreducible or not.
   (a) $x^4 + 1 \in \mathbb{R}[x]$.
   (b) $x^4 + 1 \in \mathbb{Q}[x]$.
   (c) $x^7 + 11x^3 - 33x + 22 \in \mathbb{Q}[x]$.
   (d) $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$.
   (e) $x^4 + x^3 + x^2 + x + 1 \in \mathbb{R}[x]$.
   (f) $x^3 - 7x^2 + 3x + 3 \in \mathbb{Q}[x]$.
   (g) $x^3 - 5 \in \mathbb{Z}_{11}[x]$.
   (h) $x^4 + x + 1 \in \mathbb{Z}_2[x]$.
2. Deduce proposition 12.1.1 (commonly called the rational root theorem) as a simple corollary from lemma 12.1.9 (Gauss' theorem).
3. Show that $x^n + 5x^{n-1} + 3$ is irreducible over $\mathbb{Z}[x]$ for any $n \geq 2$, by consider $\mathbb{Z}_3$. (Hint: Try to mimick the proof of Eisenstein's criterion.)
4. Let $a_1, ..., a_n$ be distinct integers, prove that $f(x) = \prod_{k=1}^{n}(x - a_i) - 1$ is irreducible in $\mathbb{Q}[x]$.
5. Given that $\mathbb{Z}[i]$ is a UFD, we will show that $f(x) = x^4 - 4x + 2$ is irreducible over $\mathbb{Z}[i][x]$ in the following exercise.
   (a) Find irreducible factors of 2, and hence explain why $f(x)$ has no linear factors in $\mathbb{Z}[i][x]$. Therefore $f(x)$ is a product of two degree 2 irreducibles if $f(x)$ was reducible.
   (b) Factorize $x^4 - 4x + 2$ into irreducible polynomials over $\mathbb{Z}_5[x]$.
   (c) Prove, or recall from Tutorial 10 that there exists a surjective ring homomorphism $\mathbb{Z}[i] \to \mathbb{Z}_5$. Deduce a contradiction from part (a) if $f(x)$ was assumed to be reducible over $\mathbb{Z}[i]$.
6. Let $p(x) \in \mathbb{Z}[x]$ be an irreducible polynomial, is $\mathbb{Z}[x]/(p)$ a field?
7. Is $\mathbb{Q}(\sqrt{2})$ isomorphic to $\mathbb{Q}(\sqrt{3})$?